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THE FUNCTIONAL EQUATION AND STRICTLY SUBSTABLE RANDOM VECTORS

BY

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Abstract. A random vector X is β-substable, $\beta \in (0, 2]$, if there exist a symmetric β-stable random vector Y and a random variable $\Theta \geqslant 0$ independent of Y such that $X \stackrel{d}{=} Y\Theta^{1/\beta}$. In this paper we investigate strictly β-substable random vectors which are generated from a strictly β-stable random vector Y. We study some of their properties. We obtain the theorem that every strictly β-stable random vector X with $\Theta \sim S_{\alpha/\beta}(\sigma, 1, 0)$ is also strictly α-stable, $\alpha < \beta$ (for the case of random variable X see, e.g., [1], [6]). The opposite theorem is also satisfied, but we obtain something more. We obtain some functional equation and we show that if a strictly β-substable random vector X is α-stable, then it has to be strictly α-stable and the mixing random variable Θ has to have a distribution $S_{\alpha/\beta}(\sigma, 1, 0)$. This is the main result of the paper.

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1. INTRODUCTION

A random variable Θ is said to have a *stable distribution* if there exist parameters $0 < \alpha \leqslant 2$, $\sigma \geqslant 0$, $-1 \leqslant \beta \leqslant 1$, and $\mu \in \boldsymbol{R}$ such that the characteristic function of Θ has the following form:

$$\boldsymbol{E}e^{it\Theta} = \begin{cases} \exp\{-\sigma^\alpha |t|^\alpha (1 - i\beta (\operatorname{sgn} t) \operatorname{tg}(\pi\alpha/2)) + i\mu t\} & \text{if } \alpha \neq 1, \\ \exp\{-\sigma |t| (1 + i\beta \cdot 2\pi^{-1} (\operatorname{sgn} t) \ln |t|) + i\mu t\} & \text{if } \alpha = 1. \end{cases}$$

We will denote the stable distribution with the above characteristic fuction by $S_\alpha(\sigma, \beta, \mu)$ and we write $\Theta \sim S_\alpha(\sigma, \beta, \mu)$.

A random variable Θ is called *strictly α-stable* if and only if $\Theta \sim S_\alpha(\sigma, \beta, 0)$ for $\alpha \neq 1$ or $\Theta \sim S_1(\sigma, 0, \mu)$. A random vector $Y = (Y_1, \ldots, Y_n)$ is *strictly α-stable* if and only if all linear combinations $\langle t, Y \rangle = \sum_{i=1}^n t_i Y_i$ are strictly α-stable random variables (see, e.g., [6], Corollary 2.4.2).

Let $0 < \alpha \leqslant 2$. It is well known (see, e.g., [6], Theorem 2.4.1) that $Y = (Y_1, Y_2, \ldots, Y_n)$ is a strictly α-stable random vector if and only if there exist a finite measure Γ on the unit sphere $S_{n-1} \subset \boldsymbol{R}^n$ and a vector $\mu^0 \in \boldsymbol{R}^n$ such that

$$Ee^{i\langle t, Y\rangle} = \exp\{-K_\alpha(t)\},$$

where

$$K_\alpha(t) = \int_{S_{n-1}} |\langle t, y\rangle|^\alpha \left(1 - i\,\mathrm{sgn}(\langle t, y\rangle)\,\mathrm{tg}\frac{\pi\alpha}{2}\right)\Gamma(dy) \quad \text{if } \alpha \neq 1,$$

$$K_1(t) = \int_{S_{n-1}} |\langle t, y\rangle| \left(1 + i\frac{2}{\pi}\mathrm{sgn}(\langle t, y\rangle)\ln|\langle t, y\rangle|\right)\Gamma(dy) - i\langle t, \mu^0\rangle$$

with the additional condition

$$\int_{S_{n-1}} \langle t, y\rangle\, \Gamma(dy) = 0.$$

In this paper we will use two theorems: one concerning characteristic functions which are analytic in a strip with the real line as a boundary, and the other stating analytic properties of characteristic functions of one-sided distributions.

THEOREM 1 ([5], Theorem 2.2.1). *Let a rectifiable Jordan curve J, in the plane of the complex variable $z = t + iv$, be such that its interior D is contained in the half-plane* $\operatorname{Im} z > 0$ *and there exists a non-degenerate real interval $I = (-\sigma, \sigma) \subseteq J$. Let g be a function of z defined and continuous on $D \cup I$, which is analytic in D. If φ, the characteristic function of the ditribution function F, coincides with g on I, then*

(i) $\beta = \liminf_{x\to\infty} [-\ln F(-x)/x] > 0$; *and*

(ii) *g is analytic in* $0 < \operatorname{Im} z < \beta$ *and admits there the representation* $g(z) = \int e^{izx} dF(x)$, *so that it has a unique extension, defined by the same integral representation, to the set* $0 \leqslant \operatorname{Im} z < \beta$, *if we impose on the extension the requirement of continuity in this set. It follows that φ is uniquely determined in this case by its restriction to I.*

In the case (ii), we shall say that *φ is analytic in* $0 < \operatorname{Im} z < \beta$, which means that there exists a function analytic in $0 < \operatorname{Im} z < \beta$, continuous in $0 \leqslant \operatorname{Im} z < \beta$ and coinciding with φ for $\operatorname{Im} z = 0$.

THEOREM 2 ([5], Theorem 2.2.2a). *Let F be a distribution function and φ its characteristic function. F is bounded to the left if and only if φ is analytic in* $\operatorname{Im} z > 0$ *and it is of exponential type there. The left extremity of F is given by*

$$\sup\{t\colon F(t) = 0\} = -\lim_{v\to\infty}[\ln\varphi(iv)/v] \quad (\textit{which exists}).$$

DEFINITION 1. A random vector $X = (X_1, \ldots, X_n)$ is *strictly β-substable*, $\beta \in (0, 2]$, if there exist a strictly β-stable random vector $Y = (Y_1, \ldots, Y_n)$ and a random variable $\Theta \geqslant 0$ independent of Y such that $X \stackrel{d}{=} Y\Theta^{1/\beta}$.

Remark 1. *If $s \geqslant 0$, then $sK_\beta(t) = K_\beta(s^{1/\beta} t)$ for every $\beta \in (0, 2]$ and $t \in \mathbf{R}^n$.*

Proof. If $\beta \neq 1$, it is obvious; if $\beta = 1$, then

$$K_1(st) = \int_{S_{n-1}} |\langle ts, y\rangle| \left(1 + i\frac{2}{\pi}\operatorname{sgn}(\langle ts, y\rangle)\ln|\langle ts, y\rangle|\right)\Gamma(dy) - i\langle ts, \mu^0\rangle$$

$$= s\left[\int_{S_{n-1}} |\langle t, y\rangle| \left(1 + i\frac{2}{\pi}\operatorname{sgn}(\langle t, y\rangle)\ln|\langle t, y\rangle|\right)\Gamma(dy)\right.$$

$$\left. - i\langle t, \mu^0\rangle + i\frac{2}{\pi}\ln s \int_{S_{n-1}} \langle t, y\rangle \Gamma(dy)\right].$$

Since $\int_{S_{n-1}} \langle t, y\rangle \Gamma(dy) = 0$, we obtain $K_1(st) = sK_1(t)$. ∎

Remark 2. *Let $\beta \in (0, 2]$. Then the characteristic function of the strictly β-substable random vector X for Y being strictly β-stable with the characteristic function $\exp\{-K_\beta(t)\}$ is given by*

$$Ee^{i\langle t, X\rangle} = E\exp\{i\langle t, Y\Theta^{1/\beta}\rangle\} = E\exp\{-K_\beta(t)\Theta\}.$$

Proof. It is enough to notice that for every $\beta \in (0, 2]$ and every non-negative random variable Θ we have $K_\beta(t\Theta^{1/\beta}) = \Theta K_\beta(t)$. Consequently,

$$Ee^{i\langle t, X\rangle} = E\exp\{i\langle t\Theta^{1/\beta}, Y\rangle\} = E\exp\{-K_\beta(t\Theta^{1/\beta})\} = E\exp\{-K_\beta(t)\Theta\}. \ ∎$$

The next theorem is satisfied for a random vector X. For the case of random variable X, see, e.g., [1], pp. 167 and 522; [6], pp. 21 and 53.

THEOREM 3. *Let $\beta \in (0, 2]$, $0 < \alpha < \beta$, and let X be a strictly β-substable random vector with the representation $X \stackrel{d}{=} Y\Theta^{1/\beta}$. If a random variable $\Theta \sim S_{\alpha/\beta}(\sigma, 1, 0)$, then X is strictly α-stable.*

Proof. Let $X \stackrel{d}{=} Y\Theta^{1/\beta}$, where $Y = (Y_1, \ldots, Y_n)$ is a strictly β-stable random vector, $\Theta \sim S_p(\sigma, 1, 0)$, $p = \alpha/\beta$, and Y and Θ are independent. Then the characteristic function of the random variable Θ has the following form:

$$\varphi(t) = \exp\left\{-\sigma^p |t|^p \left(1 - i(\operatorname{sgn} t)\operatorname{tg}\frac{\pi p}{2}\right)\right\}$$

$$= \exp\left\{-\sigma^p (t(\operatorname{sgn} t))^p \left(\cos\frac{\pi p}{2} - i(\operatorname{sgn} t)\sin\frac{\pi p}{2}\right)/\cos\frac{\pi p}{2}\right\}$$

$$= \exp\left\{-\sigma^p (t(\operatorname{sgn} t)\exp\{-i(\operatorname{sgn} t)\cdot \pi/2\})^p/\cos\frac{\pi p}{2}\right\}$$

$$= \exp\left\{-\sigma^p(-it)^p/\cos\frac{\pi p}{2}\right\},$$

since $\exp\{-i(\operatorname{sgn} t)\cdot\pi/2\} = -i(\operatorname{sgn} t)$. The support of the distribution of $\Theta \sim S_p(\sigma, 1, 0)$ for $p\in(0, 1)$ is equal to $(0, \infty)$ (see, e.g., [6], Remark 2, p. 15; [3], Theorem 5.8.3).

Let $\tilde{g}(z) = \exp\{-\sigma^p(-iz)^p/\cos(\pi p/2)\}$, $z\in \boldsymbol{C}$, $D = \{z\colon \operatorname{Im} z > 0\}$. Theorem 2 implies that there exists $g(z)$ which is a continuous function on $D\cup \boldsymbol{R}$, analytic in D, and φ coincides with g on $\boldsymbol{R}$, i.e. $\varphi(t) = g(t)$ for every $t\in \boldsymbol{R}$. Hence Theorem 1 implies that g admits the representation $g(z) = \int e^{izx}\,dF(x) = Ee^{-(-iz)\Theta}$ for $\operatorname{Im} z \geqslant 0$, and this representation is unique. Since $\tilde{g}$ is a continuous function on $D\cup \boldsymbol{R}$, analytic in D and coincides with φ on $\boldsymbol{R}$, we conclude that for $r = -iz$

$$g(ri) = \int\limits_{\boldsymbol{R}_+} e^{-rx}\,dF(x) = \exp\left\{-\sigma^p r^p/\cos\frac{\pi p}{2}\right\} = \tilde{g}(ri) \quad \text{for } \operatorname{Re} r \geqslant 0.$$

Hence $Ee^{-z\Theta} = \exp\{-Az^{\alpha/\beta}\}$ for $\operatorname{Re} z \geqslant 0$, $A = \sigma^{\alpha/\beta}/\cos(\pi\alpha/2\beta)$. Thus, using Remark 2 we have

$$Ee^{i\langle t, X\rangle} = \exp\{-A\,[K_\beta(t)]^{\alpha/\beta}\}.$$

Now, it is enough to show that such a random vector X is strictly α-stable. Without loss of generality we can assume that $A = 1$. Let $a, b > 0$ and X_1, X_2 be independent copies of X. Then Remark 1 implies that

$$\begin{aligned} E\exp\{i\langle t, aX_1+bX_2\rangle\} &= E\exp\{i\langle at, X_1\rangle\}\cdot E\exp\{i\langle bt, X_2\rangle\} \\ &= \exp\{-K_\beta(at)^{\alpha/\beta}\}\cdot\exp\{-K_\beta(bt)^{\alpha/\beta}\} \\ &= \exp\{-K_\beta(t)^{\alpha/\beta}(a^\alpha+b^\alpha)\} \\ &= \exp\{-K_\beta\big((a^\alpha+b^\alpha)^{1/\alpha}t\big)^{\alpha/\beta}\} \\ &= E\exp\{i\langle t, (a^\alpha+b^\alpha)^{1/\alpha}X\rangle\}. \end{aligned}$$

Hence the random vector $X \stackrel{d}{=} Y\Theta^{1/\beta}$ is strictly α-stable in $\boldsymbol{R}^n$ for every $\beta\in(0, 2]$ and $0 < \alpha < \beta$. ∎

From the proof of the previous theorem we obtain also the following

Remark 3. *The Laplace transform* $Ee^{-z\Theta}$, $z\in \boldsymbol{C}$, $\operatorname{Re} z \geqslant 0$, *of the random variable* $\Theta \sim S_p(\sigma, 1, 0)$, $p\in(0, 1)$, $\sigma > 0$ *is equal to* $\exp\{-\sigma^p z^p/\cos(\pi p/2)\}$.

One can show that the opposite theorem to Theorem 3 is also satisfied. We omit this theorem because it is a special case of Theorem 4 which is more general. It turns out that if a strictly β-substable random vector X is α-stable, then it has to be strictly α-stable and the mixing random variable Θ has to have the distribution $S_{\alpha/\beta}(\sigma, 1, 0)$. In the proof of this fact we will need to solve a quite difficult functional equation and it is the point of the next section.

2. THE FUNCTIONAL EQUATION

In this section we will need the fact that strictly stable distributions are cancelable. We have found only the proof in the case of symmetric stable distributions (see [7], Theorem 1). Hence, in the sequel we give the proof of this fact for nonsymmetric strictly stable random variables. In this proof we will use an idea of the set of functions separating points. We say that a set of functions $A = \{f(x)\colon x \in K\}$ *separates points* of K if for any two points $x, y \in K$, $x \neq y$, there exists a function $f \in A$ such that $f(x) \neq f(y)$.

PROPOSITION 1. *Let X be a nonsymmetric, strictly α-stable random variable, $\alpha \in (0, 2]$, and let Θ_1, Θ_2 be random variables independent of X. If $X\Theta_1 \stackrel{d}{=} X\Theta_2$, then $\Theta_1 \stackrel{d}{=} \Theta_2$.*

Proof. Let $\varphi(t) = \exp\{-K_{\alpha,\sigma}(t)\}$, where

$$K_{\alpha,\sigma}(t) = \begin{cases} \sigma^\alpha |t|^\alpha \big(1 - i\beta(\operatorname{sgn} t)\operatorname{tg}(\pi\alpha/2)\big) & \text{if } \alpha \neq 1, \\ \sigma|t| - i\mu t & \text{if } \alpha = 1, \end{cases}$$

be the characteristic function of the random variable X. It is easy to check that the function $\varphi(t)$ is an injection separately on the positive and negative half-lines, and $\varphi^r(t) = \varphi(r^{1/\alpha} t)$ for $r \geqslant 0$.

Let $\Theta_1 \sim \lambda_1$ and $\Theta_2 \sim \lambda_2$. Since $X\Theta_1 \stackrel{d}{=} X\Theta_2$, we obtain for every $r \geqslant 0$

$$\int_{-\infty}^{+\infty} \varphi^r(ts)\lambda_1(ds) = \int_{-\infty}^{+\infty} \varphi^r(ts)\lambda_2(ds), \qquad t \in \boldsymbol{R}.$$

Define now $A(\varphi) = \{\varphi^r(t)\colon r \geqslant 0\}$ and $L(\varphi) = \{\sum_i a_i \varphi^{r_i}(t)\colon a_i \in \boldsymbol{R},\, r_i \geqslant 0\}$. Let us notice that $1 = \varphi^0(t) \in A(\varphi)$ and $\lim_{t\to-\infty}\phi(t) = \lim_{t\to+\infty}\phi(t) = 0$ for every function $\phi(t) \in A(\varphi)$. Moreover, if $|x| \neq |y|$, then $\operatorname{Re}\phi(x) \neq \operatorname{Re}\phi(y)$ for $\phi \in A(\varphi)$. Hence every function $\phi \in A(\varphi)$, $\phi \not\equiv 1$, separates points with different absolute values. If $x = -y$, then we can choose σ_0 such that $\pi^{-1} R_{\sigma_0}(x) \notin \boldsymbol{Z}$, where

$$R_{\sigma_0}(x) = \begin{cases} \sigma_0^\alpha \beta |x|^\alpha (\operatorname{sgn} x)\operatorname{tg}(\pi\alpha/2) & \text{if } \alpha \neq 1, \\ \mu x & \text{if } \alpha = 1. \end{cases}$$

Putting $\phi_{\sigma_0}(t) = \exp\{-K_{\alpha,\sigma_0}(t)\} \in A(\varphi)$ we have

$$\begin{aligned} \operatorname{Im}\phi_{\sigma_0}(x) - \operatorname{Im}\phi_{\sigma_0}(y) &= \exp\{-\sigma_0^\alpha |x|^\alpha\}\sin\big(R_{\sigma_0}(x)\big) - \exp\{-\sigma_0^\alpha |y|^\alpha\}\sin\big(R_{\sigma_0}(y)\big) \\ &= \exp\{-\sigma_0^\alpha |x|^\alpha\}\, 2\sin\big(R_{\sigma_0}(x)\big) \neq 0, \end{aligned}$$

and hence $\phi_{\sigma_0}(x) \neq \phi_{\sigma_0}(y)$. Thus, the set $L(\varphi)$ separates points of the one-point compactification of the real line $\bar{\boldsymbol{R}} = \boldsymbol{R} \cup \{\infty\}$. Moreover, we have

$$\int_{-\infty}^{+\infty} \phi(ts)\lambda_1(ds) = \int_{-\infty}^{+\infty} \phi(ts)\lambda_2(ds), \qquad \phi \in L(\varphi).$$

From the Stone–Weierstrass Theorem (see [2], Theorem 4E) it follows that $L(\varphi)$ is dense in the space of continuous and bounded functions on $\bar{R}$. Hence for every continuous function f on $\bar{R}$ we have

$$\int_{-\infty}^{+\infty} f(x)\lambda_1(dx) = \int_{-\infty}^{+\infty} f(x)\lambda_2(dx),$$

and consequently $\lambda_1 = \lambda_2$. ∎

THEOREM 4. *Let $\beta \in (0, 2]$, $0 < \alpha < \beta$, and let X be a strictly β-substable random vector with the representation $X \stackrel{d}{=} Y\Theta^{1/\beta}$. If X is α-stable, then the random variable $\Theta \sim S_{\alpha/\beta}(\sigma, 1, 0)$, and consequently X is strictly α-stable.*

Proof. The fact that X is a strictly β-substable random vector implies that $X \stackrel{d}{=} Y\Theta^{1/\beta}$ for some nonnegative random variable Θ and strictly β-stable random vector Y independent of Θ.

Let X_i, Y_i, Θ_i $(i = 1, \ldots, 7)$ be independent copies of vectors X, Y and random variable Θ; moreover, let Y_i and Θ_j be independent for all i, j and let $a, b \in R_+$. Since random vectors Y_i are strictly β-stable and vectors X_i are α-stable, we obtain the following equations:

$$\begin{aligned} aX_1+bX_2+aX_3+bX_4 &\stackrel{d}{=} [(a^\alpha+b^\alpha)^{\beta/\alpha}\Theta_1+(a^\alpha+b^\alpha)^{\beta/\alpha}\Theta_2]^{1/\beta}Y_1+2d \\ &\stackrel{d}{=} [a^\beta\Theta_1+b^\beta\Theta_2+a^\beta\Theta_3+b^\beta\Theta_4]^{1/\beta}Y_1 \\ &\stackrel{d}{=} [(a^\alpha+b^\alpha)^{\beta/\alpha}\Theta_1+a^\beta\Theta_2+b^\beta\Theta_3]^{1/\beta}Y_1+d, \end{aligned}$$

where $d \in R^n$. In the first line we apply twice stability of X, and then strict stability of Y. In the second line we use only strict stability of Y. In the third line we use stability of X for first two summands, and then strict stability of Y. Hence we have

$$\begin{aligned} &[(a^\alpha+b^\alpha)^{\beta/\alpha}\Theta_1+(a^\alpha+b^\alpha)^{\beta/\alpha}\Theta_2]^{1/\beta}Y_1+2[a^\beta\Theta_3+b^\beta\Theta_4+a^\beta\Theta_5+b^\beta\Theta_6]^{1/\beta}Y_2 \\ &\stackrel{d}{=} 2[(a^\alpha+b^\alpha)^{\beta/\alpha}\Theta_1+a^\beta\Theta_2+b^\beta\Theta_3]^{1/\beta}Y_1+[a^\beta\Theta_4+b^\beta\Theta_5+a^\beta\Theta_6+b^\beta\Theta_7]^{1/\beta}Y_2. \end{aligned}$$

Using once more the fact that random vectors Y_i, $i = 1, 2$, are strictly β-stable we have

$$\begin{aligned} &[(a^\alpha+b^\alpha)^{\beta/\alpha}\Theta_1+(a^\alpha+b^\alpha)^{\beta/\alpha}\Theta_2+(2a)^\beta\Theta_3+(2b)^\beta\Theta_4+(2a)^\beta\Theta_5+(2b)^\beta\Theta_6]^{1/\beta}Y_1 \\ &\stackrel{d}{=} [2^\beta(a^\alpha+b^\alpha)^{\beta/\alpha}\Theta_1+(2a)^\beta\Theta_2+(2b)^\beta\Theta_3+a^\beta\Theta_4+b^\beta\Theta_5+a^\beta\Theta_6+b^\beta\Theta_7]^{1/\beta}Y_1. \end{aligned}$$

Let $Y_1 = (Y_1^1, \ldots, Y_n^1)$. Then for every $i = 1, 2, \ldots, n$

$$\begin{aligned} &[(a^\alpha+b^\alpha)^{\beta/\alpha}\Theta_1+(a^\alpha+b^\alpha)^{\beta/\alpha}\Theta_2+(2a)^\beta\Theta_3+(2b)^\beta\Theta_4+(2a)^\beta\Theta_5+(2b)^\beta\Theta_6]^{1/\beta}Y_i^1 \\ &\stackrel{d}{=} [2^\beta(a^\alpha+b^\alpha)^{\beta/\alpha}\Theta_1+(2a)^\beta\Theta_2+(2b)^\beta\Theta_3+a^\beta\Theta_4+b^\beta\Theta_5+a^\beta\Theta_6+b^\beta\Theta_7]^{1/\beta}Y_i^1, \end{aligned}$$

and from Proposition 1 it follows that

$$(a^\alpha+b^\alpha)^{\beta/\alpha}\Theta_1+(a^\alpha+b^\alpha)^{\beta/\alpha}\Theta_2+(2a)^\beta\Theta_3+(2b)^\beta\Theta_4+(2a)^\beta\Theta_5+(2b)^\beta\Theta_6$$
$$\stackrel{d}{=} 2^\beta(a^\alpha+b^\alpha)^{\beta/\alpha}\Theta_1+(2a)^\beta\Theta_2+(2b)^\beta\Theta_3+a^\beta\Theta_4+b^\beta\Theta_5+a^\beta\Theta_6+b^\beta\Theta_7.$$

Putting $a^\beta = c$ and $b^\beta = d$ we get

$$(c^{\alpha/\beta}+d^{\alpha/\beta})^{\beta/\alpha}\Theta_1+(c^{\alpha/\beta}+d^{\alpha/\beta})^{\beta/\alpha}\Theta_2+2^\beta(c\Theta_3+d\Theta_4)+2^\beta(c\Theta_5+d\Theta_6)$$
$$\stackrel{d}{=} 2^\beta(c^{\alpha/\beta}+d^{\alpha/\beta})^{\beta/\alpha}\Theta_1+2^\beta(c\Theta_2+d\Theta_3)+c\Theta_4+d\Theta_5+c\Theta_6+d\Theta_7.$$

Now let φ be the characteristic function of the random variable Θ. Hence we obtain the following functional equation:

(1) $$\varphi^2(ct)\varphi^2(dt)\varphi\big(2^\beta(c^{\alpha/\beta}+d^{\alpha/\beta})^{\beta/\alpha}t\big)\varphi(2^\beta ct)\varphi(2^\beta dt)$$
$$= \varphi^2\big((c^{\alpha/\beta}+d^{\alpha/\beta})^{\beta/\alpha}t\big)\varphi^2(2^\beta ct)\varphi^2(2^\beta dt), \qquad c, d > 0,\ t\in\mathbf{R}.$$

Notice that $\varphi(t) \neq 0$ for every $t\in\mathbf{R}$. Indeed, let us assume that there exists $t > 0$ such that $\varphi(t) = 0$, and let $t_0 = \inf\{t > 0\colon \varphi(t) = 0\}$. Since φ is a continuous function and $\varphi(0) = 1$, we obtain $t_0 > 0$ and $\varphi(t_0) = 0$. Substituting

$$t \to t/\big(2^\beta(c^{\alpha/\beta}+d^{\alpha/\beta})^{\beta/\alpha}\big) \stackrel{\text{def}}{=} t/(2^\beta A)$$

in the equation (1) we have

$$\varphi^2\left(\frac{t}{2^\beta\cdot A/c}\right)\varphi^2\left(\frac{t}{2^\beta\cdot A/d}\right)\varphi(t)\varphi\left(\frac{t}{A/c}\right)\varphi\left(\frac{t}{A/d}\right) = \varphi^2\left(\frac{t}{2^\beta}\right)\varphi^2\left(\frac{t}{A/c}\right)\varphi^2\left(\frac{t}{A/d}\right).$$

Taking $t = t_0$ we obtain

$$\varphi\left(\frac{t_0}{2^\beta}\right) = 0 \quad \text{or} \quad \varphi\left(\frac{t_0}{A/c}\right) = 0 \quad \text{or} \quad \varphi\left(\frac{t_0}{A/d}\right) = 0.$$

Since A/c, A/d, $2^\beta > 1$, we get a contradiction to the choice of t_0. In a similar way we proceed if there exists $t < 0$ such that $\varphi(t) = 0$; then we define $t_0 = \sup\{t < 0\colon \varphi(t) = 0\}$.

Since $\varphi(t) \neq 0$, we can transform the equation (1) into the equation:

$$\varphi^2(ct)\varphi^2(dt)\varphi\big(2^\beta(c^{\alpha/\beta}+d^{\alpha/\beta})^{\beta/\alpha}t\big) = \varphi^2\big((c^{\alpha/\beta}+d^{\alpha/\beta})^{\beta/\alpha}t\big)\varphi(2^\beta ct)\varphi(2^\beta dt).$$

Define now $h(t) = \varphi(ct)\varphi(dt)/\varphi\big((c^{\alpha/\beta}+d^{\alpha/\beta})^{\beta/\alpha}t\big)$. It is easy to check that $h(-t) = \overline{h(t)}$, $h(0) = 1$ and

$$h^2(t) = h(2^\beta t), \qquad t\in\mathbf{R}.$$

Hence

$$\ln h(t) = \tfrac{1}{2}\ln h(2^\beta t).$$

Putting $r = -1/\beta$ we can write the last equation in the following form:

$$|t|^r\ln h(t) = 2^{\beta r}|t|^r\ln h(2^\beta t).$$

Substituting $H(t) = |t|^r \ln h(t)$ we obtain

$$H(t) = H(2^\beta t).$$

Thus, $h(t) = \exp\{H(t)|t|^{1/\beta}\}$ and

$$(2) \qquad \varphi\big((c^{\alpha/\beta}+d^{\alpha/\beta})^{\beta/\alpha} t\big)\exp\{H(t)|t|^{1/\beta}\} = \varphi(ct)\varphi(dt), \qquad t \in \boldsymbol{R}.$$

From the above equation it follows that $H(-t) = \overline{H(t)}$.

Substituting $t = 1$ and $g(t^{\alpha/\beta}) = \varphi(t)\exp\{-H(1)\}$ in the equation (2) we have

$$g(c^{\alpha/\beta})g(d^{\alpha/\beta}) = g(c^{\alpha/\beta}+d^{\alpha/\beta}).$$

Let $r = c^{\alpha/\beta}$ and $s = d^{\alpha/\beta}$, $r, s > 0$. Then we obtain the following Cauchy equation:

$$g(r)g(s) = g(r+s),$$

which has only one solution $g(t) = B^t$, $B = g(1) = \varphi(1)\exp\{-H(1)\} \neq 0$, $B \in \boldsymbol{C}$. Hence

$$\varphi(t) = \exp\{H(1)\}B^{t^{\alpha/\beta}} = \exp\{H(1)+(\ln B)t^{\alpha/\beta}\}, \qquad t > 0.$$

Since $\varphi(0^+) = 1$, we have $H(1) = 0$. Hence $\ln B = \ln\varphi(1) = D$ and

$$\varphi(t) = \exp\{Dt^{\alpha/\beta}\}, \qquad t > 0.$$

Consequently, $\varphi(-t) = \overline{\varphi(t)} = \exp\{\overline{\ln\varphi(1)}\,t^{\alpha/\beta}\}$, which implies that if $t < 0$, then

$$\varphi(t) = \exp\{\overline{\ln\varphi(1)}(-t)^{\alpha/\beta}\}.$$

Thus

$$\varphi(t) = \exp\{D|t|^{\alpha/\beta}\}, \qquad t \in \boldsymbol{R},$$

where

$$D = \begin{cases} \ln\varphi(1) & \text{if } t > 0, \\ \overline{\ln\varphi(1)} & \text{if } t < 0. \end{cases}$$

Now, it is easy to see that the random variable Θ is strictly α/β-stable, and hence $\Theta \sim S_{\alpha/\beta}(\sigma, \gamma, 0)$. Since $\Theta \geqslant 0$, we obtain $\gamma = 1$ (see, e.g., [6], Remark 2, p. 15; Property 1.2.14). Theorem 3 implies that the random vector $\boldsymbol{X}$ is strictly α-stable, which completes the proof. ∎

3. SOME PROPERTIES OF STRICTLY SUBSTABLE RANDOM VECTORS

Let $\boldsymbol{X} \stackrel{d}{=} \boldsymbol{Y}\Theta^{1/\beta}$ be a strictly β-substable random vector with the random vector $\boldsymbol{Y} \sim \gamma_\beta$ and a nonnegative random variable $\Theta \sim \lambda$ independent of $\boldsymbol{Y}$. It is easy to see that for the random vector X we have

$$P\{(X_1, \ldots, X_n)\in B\} = \int_0^\infty P\{(Y_1, \ldots, Y_n)\in Bu^{-1/\beta}\}\,\lambda(du)$$

$$= \int_0^\infty \gamma_\beta(Bu^{-1/\beta})\,\lambda(du).$$

That is why the distribution of X is often called the *scale mixture* of the distribution γ_β with respect to the mixing measure λ.

Notice that we can mix the measure γ_β also with respect to a signed measure m on $[0, \infty)$ defining

$$\gamma_\beta \circ m(B) = \int_0^\infty \gamma_\beta(Bu^{-1/\beta})\, m(du).$$

This formula makes sense if the measure m is finite, but the result of such a mixture will be a signed measure. If $m(\{0\}) = p > 0$, then

$$\gamma_\beta \circ m(B) = p\delta_0(B) + \int_{(0,\infty)} \gamma_\beta(Bu^{-1/\beta})\, m(du).$$

Now we give some properties of strictly substable random vectors.

THEOREM 5. *Let X be a strictly β-substable random vector, $\beta\in(0, 2]$, and let a nonnegative random variable Θ be infinitely divisible. Then the random vector X is also infinitely divisible.*

Proof. Let $\Theta \sim \lambda$ be an infinitely divisible random variable. Then for every $n\in N$ there exists a random variable $\Theta_n \sim \lambda_n$ such that

$$\Theta \stackrel{d}{=} \Theta_{n,1} + \ldots + \Theta_{n,n},$$

where $\Theta_{n,i}$, $i = 1, \ldots, n$, are independent copies of Θ_n. Thus

$$Ee^{i\langle t,X\rangle} = \int_0^\infty \exp\{-uK_\beta(t)\}\,\lambda(du)$$

$$= \int_0^\infty \exp\{-uK_\beta(t)\}\,\lambda_n^{*n}(du) = \Big[\int_0^\infty \exp\{-uK_\beta(t)\}\,\lambda_n(du)\Big]^n$$

$$= [E\exp\{i\langle t, Y\Theta_n^{1/\beta}\rangle\}]^n = E\exp\{i\langle t, Y\Theta_{n,1}^{1/\beta}\rangle\}\cdot\ldots\cdot E\exp\{i\langle t, Y\Theta_{n,n}^{1/\beta}\rangle\}.$$

Hence

$$X \stackrel{d}{=} Y(\Theta_{n,1} + \ldots + \Theta_{n,n})^{1/\beta} \stackrel{d}{=} Y_1\Theta_{n,1}^{1/\beta} + \ldots + Y_n\Theta_{n,n}^{1/\beta},$$

where $Y_1, \ldots, Y_n$ are independent copies of Y, and Y_i and $\Theta_{n,j}$ for all $i, j = 1, \ldots, n$ are independent. ∎

Let us remind here a very well-known concept of the exponent of the measure $\boldsymbol{m}$ (possibly a signed measure) on $\boldsymbol{R}^n$:

$$\operatorname{Exp}(\boldsymbol{m}) \stackrel{\text{def}}{=} \exp\{-\boldsymbol{m}(\boldsymbol{R}^n)\} \sum_{k=0}^{\infty} \frac{1}{k!} \boldsymbol{m}^{*k}.$$

For every positive measure $\boldsymbol{m}$ on $\boldsymbol{R}^n$ it follows that $\operatorname{Exp}(\boldsymbol{m})$ is an infinitely divisible probability measure with the characteristic function

$$(\operatorname{Exp}(\boldsymbol{m}))^{\wedge}(\boldsymbol{t}) = \exp\{-\int_{\boldsymbol{R}^n} (1-e^{i\langle \boldsymbol{t},\boldsymbol{y}\rangle})\,\boldsymbol{m}(d\boldsymbol{y})\}.$$

LEMMA 1. *Let m be a finite signed measure on $[0, \infty)$, $\beta \in (0, 2]$. Then*

$$\int_0^\infty e^{-us} m^{*n}(du) = \Big[\int_0^\infty e^{-us} m(du)\Big]^n.$$

Proof. This formula does not require any proof if m is a probability measure, otherwise let $m = m^+ - m^-$. Then, obviously,

$$\lambda^+ = m^+/m^+([0, \infty)) \stackrel{\text{def}}{=} m^+/a \quad \text{and} \quad \lambda^- = m^-/m^-([0, \infty)) \stackrel{\text{def}}{=} m^-/b$$

are probability measures on $[0, \infty)$. Let $U_1 \sim \lambda^+$ and $U_2 \sim \lambda^-$. Now we need only standard calculations:

$$\begin{aligned}
\int_0^\infty e^{-us} m^{*n}(du) &= \int_0^\infty e^{-us} (a\lambda^+ - b\lambda^-)^{*n}(du) \\
&= \sum_{k=0}^{\infty} \binom{n}{k} a^k b^{n-k} (-1)^{n-k} \int_0^\infty e^{-us} (\lambda^+)^{*k} * (\lambda^-)^{*(n-k)}(du) \\
&= \sum_{k=0}^{\infty} \binom{n}{k} (-1)^{n-k} [aE_{\lambda^+} \exp\{-U_1 s\}]^k \cdot [bE_{\lambda^-} \exp\{-U_2 s\}]^{n-k} \\
&= \Big[\int_0^\infty e^{-us} m(du)\Big]^n. \ \blacksquare
\end{aligned}$$

THEOREM 6. *For every strictly β-stable measure γ_β, $\beta \in (0, 2]$, and every finite signed measure m on $[0, \infty)$*

$$\gamma_\beta \circ \operatorname{Exp}(m) = \operatorname{Exp}(\gamma_\beta \circ m).$$

Proof. Let $\boldsymbol{Y}$ be a strictly β-stable random vector with distribution γ_β and density function $f_\beta(\boldsymbol{y})$, and $\lambda = \operatorname{Exp}(m)$. Then the Fourier transform of the measure $\gamma_\beta \circ \lambda$ is equal to

$$\int_0^\infty \hat{\gamma}_\beta(\boldsymbol{t}u^{1/\beta}) \operatorname{Exp}(m)(du) = \exp\{-m(\boldsymbol{R}_+)\} \sum_{k=0}^{\infty} \frac{1}{k!} \int_0^\infty \exp\{-uK_\beta(\boldsymbol{t})\}\, m^{*k}(du).$$

Using Lemma 1 we obtain

$$\begin{aligned}
(\gamma_\beta \circ \operatorname{Exp}(m))^\wedge (t) &= \exp\{-m(R_+)\} \sum_{k=0}^{\infty} \frac{1}{k!} \Big[\int_0^\infty \exp\{-uK_\beta(t)\}\, m(du)\Big]^k \\
&= \exp\Big\{-\int_0^\infty \big(1-\exp\{-uK_\beta(t)\}\big)\, m(du)\Big\} \\
&= \exp\Big\{-\int_0^\infty \Big(1-\int_{R^n} \exp\{i\langle u^{1/\beta} t,\, y\rangle\} f_\beta(y)\, dy\Big)\, m(du)\Big\} \\
&= \exp\Big\{-\int_0^\infty \int_{R^n} \big(1-\exp\{i\langle t,\, u^{1/\beta} y\rangle\}\big) f_\beta(y)\, dy\, m(du)\Big\} \\
&= \exp\Big\{-\int_{R^n} \big(1-e^{i\langle t,v\rangle}\big) \int_0^\infty f_\beta(vu^{-1/\beta})\, u^{-n/\beta}\, m(du)\, dv\Big\} \\
&= (\operatorname{Exp}(\gamma_\beta \circ m))^\wedge (t). \ \blacksquare
\end{aligned}$$

Theorem 6 implies

Remark 4. *The strictly β-substable random vector X with mixing probability measure $\lambda = \operatorname{Exp}(m)$ for some finite signed measure m on $[0, \infty)$ is infinitely divisible if only*

$$\int_0^\infty f_\beta(tu^{-1/\beta})\, u^{-n/\beta}\, m(du) \geqslant 0 \qquad \textit{for all } t \in R^n.$$

An example of a mixing variable $\Theta \sim \operatorname{Exp}(m)$ which is not infinitely divisible while the β-substable random vector X (which is also strictly β-substable) has this property (i.e. the above condition is satisfied) can be found, e.g., in [4], p. 94.

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